14w5136 Algebraic and Model Theoretical Methods in Constraint Satisfaction

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1 Overview of the field and the structure of the workshop

The aim in a constraint satisfaction problem (CSP) is to find an assignment of values to a given set of variables, subject to constraints on the values which can be assigned simultaneously to certain specified subsets of variables. CSPs are used to model a wide variety of computational problems in computer science, discrete mathematics, artificial intelligence, and elsewhere, and they have found numerous applications in those areas. Several very successful approaches to study the complexity and algorithms for constraint satisfaction problems have been developed over the last decade. One of the most fruitful uses universal algebra. So far the bulk of research on the CSP has been done assuming the variables can take only a finite number of values. The infinite CSP, in which this restriction is removed, has much stronger expressive power, but is also much harder to study. Two recent discoveries made it possible to transfer some techniques, in particular the algebraic approach, from the finite to some classes of the infinite CSP. The first one is the expansion of the algebraic approach onto infinite domain CSPs by Bodirsky and Nešetřil. The universal algebraic approach rests on the observation that primitive positive definability in finite structures can be characterized by polymorphisms. The second one is a surprising connection between topological dynamics and Ramsey theory discovered by Kechris, Pestov, and Todorcevic [?]. This is highly relevant in the context of constraint satisfaction, since the universal-algebraic approach is particularly powerful for omega-categorical structures that are definable in ordered Ramsey structures.

In contrast to finite domain CSPs the study of infinite domain CSPs through algebra and model theory is still in its infancy, and many directions, problems, and hypotheses are yet to be identified. The workshop therefore focuses on the interaction of three areas of mathematics and computer science:

- Universal Algebra and its applications in the finite CSP,

- Infinite CSP and its algebraic theory,

- Model Theory and Topological Dynamics.

This workshop brought together for the first time researchers from these three areas to explore the implications of this new connection between them.

Along with 19 contributing 1- and 0.5-hour talks highlighted in subsequent sections, the workshop featured three tutorials corresponding to the three main areas, aiming to bring up the participants' background to

the level necessary to understand the contributing talks. The tutorials were given by M.Pinsker on "Constraint satisfaction problems on infinite domains", by P.Cameron on "Synchronization, graph endomorphisms, and some remarkable graphs", and by R.Willard on "Universal algebra and CSP".

2 Universal algebra and finite CSP

In the theoretical study of CSPs that has witnessed a rapid growth over the last decade, the principal research problems are: to determine the computational and descriptive complexity of constraint problems and to develop efficient algorithms for solving CSPs whenever such algorithms exist. The main focus of research has been the computational and descriptive complexity of finite domain CSPs, in which the set of possible values of variables is finite. The quest for a classification of CSPs restricted by a constraint language has been guided by the dichotomy conjecture posed in 1993 by Feder and Vardi [**?**]. This conjecture claims that any CSP of this type is either polynomial time solvable, or NP-complete; despite significant progress this conjecture remains unresolved. The bulk of research done towards resolving the dichotomy conjecture uses connections between the complexity of CSPs with restrictions on the allowed constraints, and concepts from universal algebra. More specifically, the complexity of CSPs is determined by the properties of polymorphisms, multi-ary operations that can be viewed as generalized symmetries of constraints. This universal-algebraic approach to CSPs has not only greatly assisted in the study of CSP complexity, but vitalizes the field of universal algebra since it raises questions that are of central importance in the study of finite algebras.

2.1 Presentation Highlights

2.1.1 Linear Datalog and k-permutability = symmetric Datalog

Alexandr Kazda reported on recent progress on characterizing low complexity CSPs. Datalog is a Prolog-like language that allows one to talk about relational structures. Given a relational structure $\mathbf{A}$, there is a Datalog program that decides CSP($\mathbf{A}$) if and only if CSP($\mathbf{A}$) can be solved by local consistency methods. Such structures are well understood. By restricting the kinds of rules a Datalog program can use, one obtains first linear and then symmetric Datalog languages. Characterizing the CSPs solvable by these fragments of Datalog is an open problem.

Kazda showed that if $\mathbf{A}$ is a finite relational structure whose algebra of polymorphisms is k-permutable for some k and CSP($\mathbf{A}$) can be solved using linear Datalog, then CSP($\mathbf{A}$) can be solved by symmetric Datalog (which is weaker). This supports the conjecture that CSP($\mathbf{A}$) is solvable by symmetric Datalog iff the algebra of polymorphisms of $\mathbf{A}$ is semidistributive and k-permutable for some k.

2.1.2 The basic CSP reductions revisited

Libor Barto revisited the classic reductions between fixed template CSPs over a finite domain. These reductions are based on three constructions on relational structures: forming images under unary endomorphisms, adding singletons to cores, and pp-interpretation. On the algebraic side, these reductions are usually interpreted in the following (or some equivalent) way: the complexity (computational or descriptive) of CSP($\mathbf{A}$) depends only on the idempotent Maltsev conditions satisfied by the core of $\mathbf{A}$. Barto offers a sharper interpretation: the complexity of CSP($\mathbf{A}$) depends only on the linear Maltsev conditions satisfied by $\mathbf{A}$.

2.1.3 Absorption and directed Jonsson terms

Ralph McKenzie improved the known characterizations of conditions on the congruence lattice through specific term operations. He proved that every congruence distributive variety has directed Jónsson terms, and every congruence modular variety has directed Gumm terms. These directed terms have the property that they respect every absorption witnessed by the original Jónsson or Gumm terms. This result is directly equivalent to a strong absorption theorem (formulated differently for CD and CM varieties.) These results were already known for locally finite varieties, but it is a surprise that they hold for unrestricted varieties as well. The absorption theorems have interesting applications in a broader domain.

The absorption theorem in question was proved by Libor Barto for finite algebras with Jonsson terms, as a main step toward his proof that a finitely related finite algebra with Jonsson terms has a near unanimity term operation. Marcin Kozik used this to prove that a finite algebra has Jonsson terms iff it has directed Jonsson terms: that is, $J_1(x, y, z), \ldots, J_n(x, y, z)$ satisfying equations $J_i(x, y, x) = x, J_1(x, x, y) = x, \ldots, J_n(x, y, y) = y$, and $J_i(x, x, y) = J_{i+1}(x, y, y)$ for $1 \leq i < n$. Both results are now extended to arbitrary (not locally finite) algebras, and shown that they are fundamentally the same result.

2.1.4 Optimal strong Maltcev conditions for congruence meet-semidistributivity

Matthew Moore provided new characterizations of locally finite, congruence meet-semidistributive varieties. These varieties have been characterized by numerous Maltcev conditions and, recently, by two strong Maltcev conditions. The new work provides three new strong Maltcev characterizations and a new Maltcev characterization each of which improves the known ones in some way.

2.1.5 Counting Matrix Partitions

Martin Dyer discussed recent work on the counting version of the Matrix Partition problem. Matrix partitions are a generalisation of graph homomorphisms. They were introduced into the context of structural graph theory, and are equivalent to homomorphisms of trigraphs, as introduced by Chudnovsky. The decision version of this problem was investigated in [**?**].

2.1.6 Descriptive Complexity of approximate counting CSPs

Victor Dalmau reported on several results related to the descriptive complexity of counting CSPs. Motivated by Fagin's characterization of NP, Saluja et al. [**?**] have introduced a logic based framework for expressing counting problems. In this setting, a counting problem (seen as a mapping C from structures to non-negative integers) is 'defined' by a first-order sentence ϕ if for every instance A of the problem, the number of possible satisfying assignments of the variables of ϕ in A is equal to $C(A)$. The logic $RH\Pi_1$ has been introduced by Dyer et al. [**?**] in their study of the counting complexity class #BIS. The interest in the class #BIS stems from the fact that, it is quite plausible that the problems in #BIS are not #P-hard, nor they admit a fully polynomial randomized approximation scheme. In this talk we shall present some results concerning the definability of counting constraint satisfaction problems #CSP(H) in the monotone fragment of $RH\Pi_1$.

2.1.7 The other side of the CSP

Hubie Chen studied the CSPs restricted by the left-hand side, formulated as the relational homomorphism problem over a set of structures $\mathbf{A}$, wherein each instance must be a pair of structures such that the first structure is an element of $\mathbf{A}$. He presented a comprehensive complexity classification of these problems, which strongly links graph-theoretic properties of $\mathbf{A}$ to the complexity of the corresponding homomorphism problem. In particular, he defined a binary relation on graph classes and completely describe the resulting hierarchy given by this relation. This binary relation is defined in terms of a notion which is called graph deconstruction and which is a variant of the well-known notion of tree decomposition. Then he used this graph hierarchy to infer a complexity hierarchy of homomorphism problems which is comprehensive up to a computationally very weak notion of reduction, namely, a parameterized form of quantifier-free reductions. He obtained a significantly refined complexity classification of left-hand side restricted homomorphism problems, as well as a unifying, modular, and conceptually clean treatment of existing complexity classifications, such as the celebrated classifications by Grohe et al. [**?**, **?**, **?**].

2.1.8 Constant-factor approximable finite-valued CSPs

Andrei Krokhin presented new results on the approximability of minimization problems related to CSP, namely the (finite-)valued constraint satisfaction problems (VCSPs) and their special case, the minimum constraint satisfaction problems (Min CSPs), all with a fixed finite constraint language Γ. The main focus is on characterising such problems that admit a constant-factor approximation algorithm.

A recent result of Ene et al. says that, under a mild technical condition, the basic LP relaxation is optimal for constant-factor approximation for VCSPs unless the Unique Games Conjecture fails. Krokhin showed that the characterisation problem for VCSPs reduces to the one for Min CSPs, and then used the algebraic approach to the CSP to characterise constraint languages such that the basic LP has a finite integrality gap for the corresponding Min CSP. We also show how this result can in principle be used to round solutions of the basic LP relaxation, and how, for several examples that cover all previously known cases, this leads to efficient constant-factor approximation algorithms. Finally, we improve the above mentioned UG-hardness of constant-factor approximation to NP-hardness for a class of Min CSPs.

2.1.9 Necessary Conditions for Tractability of Valued CSPs

Stanislav Zivny gave a talk on a Galois connection for weighted clones. The connection between constraint languages and clone theory has been a fruitful line of research on the complexity of constraint satisfaction problems. In a recent result, Cohen et al. [?] have characterised a Galois connection between valued constraint languages and so-called weighted clones. In his talk Zivny studied the structure of weighted clones. We extend the results of Creed and Zivny from [?, ?] and provide necessary conditions for tractability of weighted clones and thus valued constraint languages. We demonstrate that some of the necessary conditions are also sufficient for tractability, while others are provably not.

2.2 Open Problems

- Are the following problems decidable?

  Given operations $f_1, \ldots, f_k$ and a relation R on a finite set A:

  1. Is the clone generated by $f_1, \ldots, f_k$ equal to the clone of functions preserving R, i.e., is $\mathrm{Clo}(f_1, \ldots, f_k) = \mathrm{Pol}(R)$?

  2. Is the clone generated by $f_1, \ldots, f_k$ finitely related, i.e., is $\mathrm{Clo}(f_1, \ldots, f_k) = \mathrm{Pol}(S)$ for some relation S on A?

  3. Is the clone of R-preserving operations finitely generated, i.e., is $\mathrm{Pol}(R) = \mathrm{Clo}(g_1, \ldots, g_l)$ for some $l \in \mathbb{N}$ and operations $g_1, \ldots, g_l$ on A?

  4. Is there a finite bound on the essential arities of operations in $\mathrm{Pol}(R)$?

- Let A be an (infinite) algebra that has a n-ary WNU, does A have to have a Taylor operation of low arity? Ie. does there exist a sequence of algebras $A_1, A_2, \ldots$ such that i has an i-ary WNU, but the minimum arity of a Taylor term in A_i tends to infinity as i increases?

- Characterise the class of relational structures A such that $CSP(A)$ can be solved by linear Datalog.

- Given a finite idempotent algebra (by tables of its operations), is it possible to decide in polynomial time whether the algebra admits a minority operation?

- One very interesting open problem is surjective CSP on the template with universe $\{1, 2, 3\}$ and relation $\{(a, b, c) \mid |\{a, b, c\}| \leq 2\}$.

  It appears in the survey on surjective homomorphism problems by Bodirsky, Kara, and Martin [?].

3 Ramsey Theory and Model Theory

The increasing emphasis on CSPs with infinite domains (see the next section), and in particular on CSPs whose domain is an ω-categorical first order structure, has led to close interactions with parts of model theory, Ramsey theory, permutation group theory (and extensions to transformation monoids), and topological dynamics. One goal, for infinite structures which give natural CSPs, is to classify the CSPs for templates which are interpretable in them. This takes various forms. A first goal might be to classify the *reducts* of a given template M up to first-order interdefinability, or equivalently to classify the closed supergroups of the automorphism group. A classification of reducts of M up to positive-primitive interdefinability corresponds

to classifying the clones (closed topologically in the clone of all polymorphisms of the *set M*) which contain the clone of polymorphisms of M.

Approaches to these problems, in recent papers of Bodirsky, Pinsker, and co-authors, mostly use structural Ramsey theory for classes of finite relational structures, as developed by Nešetřil, Rödl and others since the 1970s. A class $\mathcal{C}$ of finite structures is a *Ramsey class* if for an $A \leq B$ in $\mathcal{C}$ and positive integer k, there is $D \in \mathcal{C}$ such that for any function f from the collection of copies of A in D to $\{1, \ldots, k\}$, there is a copy B' of B in D such that f is constant on the copies of A in B'. Under reasonable conditions, if $\mathcal{C}$ is a Ramsey class then there is a homogeneous structure (a countably infinite structure such that any isomorphism between finite substructures extends to an automorphism) whose age (the collection of finite structures which embed in it) equals $\mathcal{C}$. This subject has major connections to a further current topic, topological dynamics (the study of continuous actions of topological groups on compact spaces). In particular, if G is a closed permutation group on a countable set M, then G is *extremely amenable* (i.e. every continuous action on a compact space has a fixed point) if and only if G is the automorphism group of a homogeneous totally ordered structures on M whose age is a Ramsey class. This equivalence (in both directions) has been used in results classifying reducts.

The meeting included a 3-lecture tutorial by Pinsker describing the connections of this field to CSPs (see Section 4), a somewhat related 3-lecture tutorial by Cameron on synchronisation and transformation monoids, and several other talks.

3.1 Presentation Highlights

3.1.1 Synchronization, graph endomorphisms, and some remarkable graphs (tutorial)

A transformation monoid on a set (here assumed finite) is *synchronizing* if it contains an element of rank 1, that is, an element whose image has cardinality 1. This tutorial by Cameron on synchronizing monoids and permutation groups described recent work, motivated ultimately by a conjecture of Cerny on reset words in finite state automata, on a topic with some connection to robot design. A permutation group G on a finite set Ω *synchronizes* the map $f : \Omega \to \Omega$ if the transformation monoid $\langle G, f \rangle$ is synchronizing. A key goal is to characterize synchronizing permutation groups.

Given a transformation monoid M on set X, the graph of M, denoted $\mathrm{Gr}(M)$, has vertex set X, with vertices v, w adjacent if and only if there is no $f \in M$ with $f(u) = v$. Then $M \leq \mathrm{End}(\mathrm{Gr}(M))$, and furthermore $\mathrm{Gr}(\mathrm{End}(\mathrm{Gr}(M))) = \mathrm{Gr}(M)$. Useful information is encoded by $\mathrm{Gr}(M)$ – for example, the minimal rank of an element of M equals the clique number and chromatic number of $\mathrm{Gr}(M)$. Cameron used $\mathrm{Gr}(M)$ to give a characterization of synchronizing transformation monoids. He noted a result of Rystsov that a permutation group G on a set of size n is primitive (that is, preserves no proper non-trivial equivalence relation on X) if and only if it synchronizes every map of rank $n-1$. Thus, every synchronizing permutation group on a set of size at least 3 is primitive (and every 2-transitive permutation group is synchronizing). The converse is false, but Araújo has conjectured that every primitive permutation group is *almost synchronizing*, i.e. synchronizes every non-uniform map (map with fibres not all of the same size).

Cameron described very recent work with Araújo and Betz around this conjecture, showing for example that every primitive group of rank at least n synchronizes every map of rank at least $n - 4$, and every non-uniform map of rank at most 4. He described some remarkable recent counterexamples to Araújo's conjecture, and gave modified conjectures and related results.

3.1.2 Simple homogeneous structures

Koponen described recent work on homogeneous structures over a finite relational language which have the model-theoretic property of *simplicity*, a combinatorial condition defined by Shelah which guarantees that there is a notion of independence with strong properties. A strengthening of simplicity is supersimplicity, and there are open questions whether every ω-categorical supersimple structure has finite rank (SU-rank, akin to a dimension), and whether every simple structure homogeneous over a finite relational language is supersimple. Koponen sketched recent proofs of these conjectures for structures which are homogeneous over a finite *binary* language. He described several further results (some joint with Ahlman) on binary homogeneous structures under strong extra assumptions on the behaviour of independence, and posed several problems.

3.1.3 Classifying homogeneous structures

Cherlin described a range of classification results and conjectures, from the last 40 years, on homogeneous structures. He gave an overview of the deep structural results of Lachlan on finite homogeneous structures over a finite relational language, and of the classifications of homogeneous graphs (Lachlan-Woodrow) and digraphs (Cherlin), homogeneous ordered graphs (Cherlin), and a recent conjecture on finite homogeneous binary structures. He also described a conjectured classification of metrically homogeneous graphs (graphs which become homogeneous when binary predicates are added to express distance) and gave evidence for this conjecture.

3.1.4 Ramsey classes with algebraic closure and forbidden homomorphisms

Hubicka gave an overview of Ramsey classes, and focussed his talk on the question: does every structure homogeneous over a finite relational language have a *Ramsey lift* (an expansion to a structure whose age is a Ramsey class), and more generally whether certain classes of finite structures have a 'Ramsey lift'. For example, by 2014 work this is known for each of the $2^{\aleph_0}$ homogeneous digraphs. He described several recent results, some joint with Nešetřil, showing that certain structures have Ramsey lifts. Part of the talk was motivated by certain ω-categorical structures identified by Cherlin, Shelah, and Shi (1998), with an associated locally finite notion of algebraic closure.

3.1.5 Continuity of homomorphisms to the clone of projections

Pongracz gave an introduction to (topological) clones of polymorphisms of ω-categorical structures. A key result of Bodirsky and Pinsker says that if Γ is an ω-categorical structure, then a structure Π is primitive positive interpretable in Γ if and only if there is a continuous clone homomorphism from the polymorphism clone $\mathrm{Pol}(\Gamma)$ to $\mathbf{1}$, the trivial clone consisting only of projections. Pongracz described several recent results in this area, for example joint work with Bodirsky and Pinsker showing that if Γ is homogeneous over a finite relational language, then if there is a homomorphism from a closed 'canonical' clone of polymorphisms of Γ (containing $\mathrm{Aut}(\Gamma)$) to $\mathbf{1}$ then there is a continuous one. This gives a dichotomy result for closed canonical clones.

3.2 Open Problems

- Find new interesting permutation groups G on an infinite set X which, for some $k \geq 4$, are k-transitive but not $(k+1)$-transitive. It is easy to give Fraïssé constructions with such automorphism groups, but there are open test questions. Is there a homogeneous relational structure with 4-transitive not 5-transitive automorphism group such that the age of M has no infinite antichains? Or such that $f(n)$, the number of isomorphism types of n-element substructures, grows no faster than exponentially? Or such that the first order theory of M does not have the 'independence property'? There are Fraïssé constructions of 3-transitive not 4-transitive permutation groups with all these properties (though the known examples are all somehow linked).

  There is evidence that the 3-4 boundary is significant. For example there are sharply 3-transitive infinite permutation groups, but, by a theorem of M. Hall, no sharply 4-transitive ones. These questions are linked to that of Pongracz below.

- For $n = 4$, is there an ω-categorical structure Δ_n with n-transitive automorphism group, such that $\mathrm{Aut}(\Delta_n)$ has a proper subflow in its action on the space LO of all total orderings of Δ_n – i.e. can there be a closed $\mathrm{Aut}(\Delta_n)$-invariant proper non-empty subset of LO? If yes, can this be done for all n? Can it be done so that Δ_n is Ramsey?

- Given finite tournaments $T_1, \ldots, T_k$, is there a decision procedure to tell whether there is an infinite antichain (under embeddability) in the collection of all finite tournaments which do not embed any of $T_1, \ldots, T_k$? The corresponding problem for permutation patterns has also been investigated.

- Suppose that M is a homogeneous but not binary relational structure. Which if any of the following implications are true?

$$M \text{ is simple} \Rightarrow M \text{ is supersimple} \Rightarrow M \text{ has finite SU-rank.}$$

- Suppose that M is a (binary) homogeneous (super)simple relational structure (with finite SU rank) (extra assumptions in parentheses optional). Can such a structure M interpret a non-trivial pregeometry?

- Suppose M is binary, homogeneous, primitive, simple, and 1-based. Must M be a binary *random* structure? What if we remove '1-based'?

- It appears that in the known examples of a set K of finite relational structures with a 0–1 law and ω-categorical almost sure theory T_K, T_K is homogenizable and simple. Are there reasonable conditions on K which guarantee this?

- Which homogeneous/homogenizable (simple) structures M can be 'constructed' as probabilistic limits of finite structures?

4 Infinite-Domain CSPs

In infinite domain constraint satisfaction, there are two major directions: the first is the generalisation of the universal-algebraic approach to finite domain CSPs to large classes of CSPs with infinite domains. The natural context here is the class of ω-categorical structures, since for those structures the complexity of the CSP is captured by the polymorphisms of the template.

The second major direction is the study of CSPs for templates that are of general interest in computer science and mathematics, which might not be ω-categorical. A typical example is the feasibility problem for linear programs over $\mathbb{R}$ or the max-atoms problem (which is polynomial-time equivalent to mean payoff games, a notorious problem of open computational complexity).

Michael Pinsker's tutorial addressed the first of those two directions, providing a detailed introduction to the universal-algebraic approach to CSPs for ω-categorical templates. Of direct relevance for this approach are the results that have been presented by Barto, and by Pongracz, reported above. But also the Ramsey lift questions treated in the talk of Hubicka are important in this context. The second direction was also well represented by talks of Barnaby Martin and Johan Thapper.

4.1 Presentation Highlights

4.1.1 Pinsker

The introduction of Michael Pinsker to the universal-algebraic approach to infinite-domain CSPs had three parts:

- In the first part he first described why the complexity of the CSP for ω-categorical templates only depends on the polymorphism clone of the template, introducing the necessary model theoretic background. He illustrated the class of problems that can be modelled in this way by graph satisfiability problems.

- In the second part Michael took a further abstraction step, considering the clone of polymorphisms only as a topological clone, and finally as an abstract clone. For ω-categorical structures, the topological polymorphism clone is closely related to the pseudovariety generated by the polymorphism algebra, and he explained how this link becomes important for proving NP-hardness of CSPs. In fact, the complexity of an ω-categorical CSP only depends on the topological polymorphism clone. In this part, Pinsker also covered an important tool for studying automorphism groups and endomorphism monoids, namely Ramsey theory.

- In the third and final part, Pinsker treated model-complete cores, which are the ω-categorical pendants to cores of finite structures. Having all these concepts available, he could present a precise formulation of a dichotomy conjecture that holds for a very broad class of ω-categorical structures. He closed with other open problems that could be relevant to solve this dichotomy conjecture.

4.1.2 Martin

Barnaby Martin presented a classification of the computational complexity of CSPs where the template has domain $\mathbb{Z}$, and where all relations are first-order definable over $(\mathbb{Z}; \mathrm{succ})$, that is, the integers with the successor function. The complexity classification is complete under the assumption that the Feder-Vardi conjecture is true. The motivation to study reducts of $(\mathbb{Z}; \mathrm{succ})$ is that this structure is probably the most fundamental structure with a finite relational signature that is *not* ω-categorical.

The surprising message of Martin's talk is that the universal-algebraic approach can be adapted even to this setting. Even though that we do not have an *a priori* argument that polymorphisms capture the computational complexity of the CSP in this setting, it turns out that, *a posteriori*, having obtained the classification, we see that this is the case. Indeed, the tractable cases are described by polymorphisms of the template, or a saturated extension of the template, that satisfy certain equations, as in the case of finite domain CSPs.

4.1.3 Thapper

Johan Thapper reported on joint work with Peter Jonsson which investigates polynomial-time tractable variants of the linear program feasibility problem. More specically, they looked at CSPs whose template contains the relation defined by $x + y = z$, and an arbitrary finite set of semilinear relations. For a large subclass of these templates, Thapper and Jonsson identified the boundary between tractable and hard CSPs. To handle the new tractable problems, they introduced a notion of affine consistency and an accompanying algorithm that can be used to decide satisfiability.

4.2 Open Problems

- Is every homomorphism from a polymorphism clone to the clone of projections $\mathbf{1}$ continuous with respect to the topology of pointwise convergence?

- Can every homogeneous structure be expanded by finitely many relations such that the resulting structure is still homogeneous and additionally Ramsey?

- Classify $\mathrm{CSP}(\Gamma)$ for all reducts Γ of the universal homogeneous permutation.

- Classify the complexity of all problems in MMSNP$_2$. The logic MMSNP is *monotone monadic SNP*, a fragment of existential second-order logic closely related to finite domain CSP. MMSNP$_2$ is the generalisation of this logic by restricted existential quantification (à la Courcelle) over higher-ary relations. All the problems in this class correspond to CSPs with ω-categorical templates.

- Does every semi-algebraic extension of linear program feasibility which is not semilinear simulate a sums-of-roots problem?

- Do CSPs for reducts of $(\mathbb{Z}; \mathrm{succ}, y = 2x)$ have a non-dichotomy?

- Do CSPs for reducts of $(\mathbb{Z}; 0, \mathrm{succ})$ have a non-dichotomy?

5 CSP Related Areas

The methods from the Constraint Satisfaction problem, universal algebra and model theory have also been used in numerous applications some of which have also been presented at the workshop.

5.1 Further Presentation Highlights

5.1.1 Dvorak

Zdenek Dvorak studied the restriction of Boolean CSPs where the incidence graph of the constraints is *planar*. This restriction on the input instances can make some NP-hard Boolean CSPs easy, for instance the problem *Not-all-equal 3-SAT*. When the template is not preserved by the map $x \mapsto -x$, hard CSPs remain hard. Otherwise, some additional CSPs become feasible via algorithms for finding maximal matchings. Dvorak

presented a precise conjecture about the border between NP-hardness and tractability in this setting: if the CSP is 'matching-realizable', it is in P, otherwise NP-complete. The conjecture has been verified for all templates with relations of arity at most five.

5.1.2 Osssona de Mendez

Patrice Osssona de Mendez presented joint results with Jarik Nešetřil on restricted dualities: a class $\mathcal{C}$ has *all restricted dualities* if every connected structure F has a *dual* D for $\mathcal{C}$, that is, F does no homomorphically map to D, and every $G \in \mathcal{C}$ maps to D if and only if F does not map to G. De Mendez and Nešetřil showed that every graph class with bounded expansion has all restricted dualities. For the case when $\mathcal{C}$ is the class of positive instances of a CSP, they found a necessary and sufficient condition for having all restricted dualities.

5.1.3 Algebraic Algorithms for the Inference Problem in Propositional Circumscription

Michal Wrona reported on new algorithms for the Inference problem. Circumscription, introduced by McCarthy [?], is perhaps the most important formalism in nonmonotonic reasoning. The inference problem for propositional circumscription in multi-valued logics may be defined in constraint-based way as follows. Let $(D; \leq)$ be a partial order on domain D and Γ a constraint language over D. An instance of the general minimal constraint inference problem $(\mathrm{GMININF}(\Gamma, (D; \leq)))$ is a set of constraints C over variables V and relations in Γ, a partition of V into three sets of variables: P that are subject to minimizing, Q that must maintain the fixed value and Z that can vary, and a constraint ψ over domain D. The question is whether every minimal solution to C is a solution to ψ.

The classification of the complexity of four variants of this problem where $D = \{0, 1\}$ and $0 < 1$ (propositional circumscription in Boolean logic) has been completed in [?]. Each version of GMININF exhibit a trichotomy among Π_2^P-complete, coNP-complete and problems solvable in polynomial time. In two versions: the most general one $(\mathrm{GMININF}(\Gamma, (D; \leq)))$ and $\mathrm{VMININF}(\Gamma, (D; \leq))$, where Q is always $\emptyset$, the complexity is fully captured by the clone of polymorphisms of Γ. To the best of our knowledge, in the full generality, GMININF has been studied so far only in [?] which focuses on the complexity dichotomy between Π_2^P-complete problems and those contained in coNP for the three element domain. The question of whether there are any interesting polynomial classes of languages is left open. We answer this question affirmatively by providing three such classes of languages for GMININF and three for VMININF for arbitrarily large finite domains. These classes fully generalize two-element tractable classes from [DHN12] and are defined by closures under certain polymorphisms. Therefore the algorithms we provide for them are fully algebraic. We believe that it is the first but serious step towards obtaining trichotomies for GMININF and VMININF for restricted cases such as the three element domain or conservative languages.

6 Comments of participants, and further research

Comments on the meeting were received afterwards from the participants, all positive about the stimulation of the meeting, the scientific organisation, and the hosting by BIRS and the Banff Centre. Several mentioned further projects initiated at the workshop. For example, Cameron and Hartman began a project on homogenizability of line graphs. Through suggestions of Dalmau, Dvorak made progress on the classification problem for planar Boolean CSPs. Pinsker made progress on a project with Barto, and with Goldstern solved a problem which will enable them to finish a paper with Shelah. Others mentioned new interactions, new problems which caught their attention for future work, and the general value of the workshop in breaking down barriers between fields (universal algebra, CSPs, combinatorics, model theory).

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